

Unitary Minimal Liouville Gravity and Frobenius Manifolds

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Abstract

We study unitary minimal models coupled to Liouville gravity using Douglas string equation. Our approach is based on the assumption that there exist an appropriate solution of the Douglas string equation and some special choice of the resonance transformation such that necessary selection rules of the minimal Liouville gravity are satisfied. We use the connection with the Frobenius manifold structure. We argue that the flat coordinates on the Frobenius manifold are the most appropriate choice for calculating correlation functions. We find the appropriate solution of the Douglas string equation and show that it has simple form in the flat coordinates. Important information is encoded in the structure constants of the Frobenius algebra. We derive these structure constants in the canonical coordinates and in the physically relevant domain in the flat coordinates. We find the leading terms of the resonance transformation and express the coefficients of the resonance transformation in terms of Jacobi polynomials.

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1 Introduction

There are two independent approaches to 2D Liouville gravity [1]. The first method is based on the continual integral over Riemann metrics. After gauge fixing, it leads to the Liouville theory interacting with matter fields. If matter fields are represented by some minimal CFT model, then this theory is called minimal Liouville gravity (MLG). The main object of study in MLG is correlation functions, which are given by integrals over moduli of curves (with punctures) from differential forms constructed through conformal blocks [2]. Another approach to 2D gravity is based on the discrete approximation of 2D surfaces by graphs (studied in the double scaling limit). This approach is known as matrix models (MM) of 2D gravity [3–9].

One of the most important results concerning 2D gravity is the coincidence of the spectra of gravitational dimensions in the MLG and MM approaches. This observation led to the hypothesis that these two approaches are equivalent [9], [10]. But the connection is not obvious, because the results for correlation functions do not coincide. As pointed out in [11], the problem in establishing the correspondence between MM and MLG is due to the freedom to add contact terms to the OPE. This effect leads to mixing the coupling constants. To identify the operators and the correlation functions, an explicit form of the resonance relations between couplings is needed. In [12], the relation between Liouville gravity and MM was studied for the Lee–Yang series of minimal models $(2, 2p+1)$. The explicit form of the resonance relations in terms of Legendre polynomials was found. In [13], these results were generalized to $(3, 3p + p_0)$. It was shown that in this case, the basic requirements, i.e., absence of vacuum expectation values of physical operators, diagonal structure of the two-point correlators, and satisfaction of the conformal fusion rules, also leads to agreement between the two approaches on the level of the correlation functions.

The technique used in [12], [13] was to some extent specific to the particular cases under consideration. It was essentially based on the requirement that one of the parameters is small ($q = 2, 3$). On the other hand, in [13], the connection of (p, q) critical points of MLG with the Frobenius manifold structure was significantly clarified. It was shown that the generating function of correlation numbers represents the logarithm of the tau-function of the integrable hierarchy related to some special Frobenius manifold. This connection can be used to obtain an explicit representation of the free energy in terms of the tau-function. The construction essentially requires knowing the structure constants of the Frobenius algebra. With these achievements, it is natural to tackle more general examples of MLG.

In this work, we use the Douglas approach and the abovementioned connection with the Frobenius manifold structure to study $(q + 1, q)$ unitary models of MLG. The rest of the paper is organized as follows. In Section 2, we briefly review the essence of the Douglas approach. We then introduce the Frobenius manifolds and find the solution of the Douglas string equation relevant for MLG. In this section, we also formulate the results

for the structure constants of the Frobenius algebra. In Section 3, we study conformal selection rules and the resonance relations. We apply the idea about the appropriate solution of the string equation formulated in the preceding section for one- and two-point correlation functions and discuss the resulting form of the resonance relations. Section 4 is for some concluding remarks.

2 Douglas approach and Frobenius manifolds

The MM of 2D gravity and the Douglas string equation approach [14] are widely studied in the literature (see, e.g., [11] and the references therein). Below, we briefly review the ideas relevant to our discussion.

The MM exhibits an infinite set of multi-critical points. In the scaling limit near the (p, q) critical point, the singular part of the free energy \mathcal{F} of the p -matrix model is constructed in terms of the solution of the string equation

$$[\hat{P}, \hat{Q}] = 1, \quad (2.1)$$

where \hat{P} and \hat{Q} are two differential operators,

$$\hat{Q} = \left(\frac{d}{dx}\right)^q + \sum_{\alpha=1}^{q-1} u_{\alpha}(x) \left(\frac{d}{dx}\right)^{q-\alpha-1}, \quad (2.2)$$

$$\hat{P} = \frac{p+q}{q^2} \hat{Q}_+^{\frac{p}{q}} + \sum_{\alpha=1}^{q-1} \sum_{k=1}^{q-1} \frac{kq+\alpha}{q^2} t_{k,\alpha} \hat{Q}_+^{k+\frac{\alpha}{q}-1}, \quad (2.3)$$

and the plus sign means that only nonnegative powers of d/dx are taken in the series expansion of the pseudo-differential operator. The constants $t_{k,\alpha}$ are usually called “times”. This choice of the basis in the parameter space is called the KdV frame.

The solution $u_{\alpha}^*(x)$ of the string equation (2.1) is related to the nonsingular part of the free energy

$$\frac{\partial^2 \mathcal{F}}{\partial x^2} = u_1^*(x). \quad (2.4)$$

The main conjecture about the connection between the Matrix Models of 2D gravity and the continuous Liouville approach is that for the appropriate solution of the string equation the singular part of the free energy \mathcal{F} coincides with the generating function of correlators of different topologies in the (p, q) MLG.¹ One further point is worthy of note. It may seem from the above simple definition that this construction comprises exhaustive information about the theory and that the remaining questions are quite technical and straightforward. But the problem is that if the partition function is the subject of the Douglas string equation, then there still remains much ambiguity related to the choice of

¹The generating function can be thought of as a partition function of the perturbed MM coupled to Liouville gravity, which is why the generating function is called sometimes the partition function.

the point in the space of theories (the point in the parameter space) and the choice of the appropriate coordinates in this space such that this particular setup leads to correct answers for the correlation functions. The transformation from the initial coordinates $t_{k,\alpha}$ to the appropriate “Liouville” coordinates λ_{mn} is known as a resonance transformation [11] (it is discussed in detail below). Hence, there is a conceptual question of clarifying the meaning of these choices as well as the meaning of the other possible solutions of (2.1) and the relations between them.

In what follows we restrict our attention to the planar limit corresponding to the spherical topology. In this case, we are dealing with the dispersionless limit of the string equation when d/dx is replaced with the variable y and the commutator is replaced with the Poisson bracket. In particular, the operator \hat{Q} transforms to the polynomial

$$Q(y) = y^q + u_1 y^{q-2} + u_2 y^{q-3} + \cdots + u_{q-1}. \quad (2.5)$$

In the planar limit, we are left with the system of first-order PDEs and the Douglas string equation takes the simple form [13], [15]

$$\frac{\partial S}{\partial u_k} = 0, \quad (2.6)$$

where the so-called action is

$$S[u_\alpha(x)] = -\operatorname{res}_{y=\infty} \left(Q^{\frac{p+q}{q}} + \sum_{1 \leq n \leq m \leq q-1} t_{mn}(\mu, \lambda) Q^{\frac{pm-qn}{q}} \right). \quad (2.7)$$

The singular part of the partition function is expressed [13] in terms of (2.7) as an integrated (closed) one-form

$$\mathcal{F} = \frac{1}{2} \int_0^{\mathbf{u}^*} \tilde{C}_k^{ij} \frac{\partial S}{\partial u^i} \frac{\partial S}{\partial u^j} du^k, \quad (2.8)$$

where \mathbf{u}^* is a suitable solution of the string equation (2.6) and \tilde{C}_k^{ij} are the structure constants² of the Frobenius algebra discussed in the next section. Hence, according to our basic idea, this function gives the generating function of (p, q) MLG

$$\mathcal{F} = \langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \rangle, \quad (2.9)$$

where the Liouville coupling constants $\lambda_{m,n}$ are expressed in terms of the times t_{mn} via the resonance transformation. Using the Douglas approach to solve MLG thus requires three main ingredients: the appropriate solution of the string equation, the resonance relations between times in the KdV frame and Liouville couplings, and, finally, the structure constants of the Frobenius algebra.

²We reserve C without the tilde for the flat coordinate system.

2.1 Frobenius algebra structure constants

In general, a Frobenius algebra A is a commutative algebra with unity equipped with a nondegenerate invariant pairing $A \otimes A \rightarrow \mathbb{C}$. The invariance means that for any three vectors a, b, c in A :

$$(a \cdot b, c) = (a, b \cdot c). \quad (2.10)$$

The Frobenius manifold M [16] is defined as a Riemannian manifold with the flat metric $g_{\mathbf{v}}$ compatible with the additional structure of the Frobenius algebra A associated with each point \mathbf{v} (or with each tangent space $T_{\mathbf{v}}M$). Let v^1, \dots, v^n be the flat coordinates on an n -dimensional Frobenius manifold M . The algebra A is identified with the tangent space $T_{\mathbf{v}}M$ by

$$e_{\alpha} \leftrightarrow \frac{\partial}{\partial v^{\alpha}}, \quad (2.11)$$

where α runs from 1 to n . We hence have the following multiplication rule for the tangent vectors at any point $\mathbf{v} \in M$:

$$\frac{\partial}{\partial v^{\alpha}} \cdot \frac{\partial}{\partial v^{\beta}} = c_{\alpha\beta}^{\gamma}(\mathbf{v}) \frac{\partial}{\partial v^{\gamma}}. \quad (2.12)$$

The axiom

$$\frac{\partial}{\partial v^{\delta}} c_{\alpha\beta\gamma}(\mathbf{v}) = \frac{\partial}{\partial v^{\gamma}} c_{\alpha\beta\delta}(\mathbf{v}) \quad (2.13)$$

defines the structure of the Frobenius manifold (see, e.g., [13] for more details about properties of Frobenius manifolds).

In Liouville gravity, we deal with a particular example of the Frobenius manifold where the corresponding algebra is defined as an algebra of polynomials $C[y]$ modulo the polynomial $Q' = dQ/dy$, where Q is defined in (2.5). The parameters u_i in (2.5) are the coordinates on the Frobenius manifold. We call these coordinated canonical coordinates. In the basis e_1, \dots, e_{q-1} , where $e_k = y^{q-1-k}$, the algebra looks like

$$e_i e_j = \tilde{C}_{ij}^k e_k \mod Q', \quad (2.14)$$

and the metric g_{ij} is given by

$$g_{ij} := \operatorname{res}_{y=\infty} \frac{e_i e_j}{Q'}. \quad (2.15)$$

The structure constants in (2.14) are exactly those that appeared in (2.8). The metric g_{ij} (and its inverse) is used to raise and lower the indices. Using this, we find

$$\tilde{C}_{ijk} = \operatorname{res}_{y=\infty} \frac{e_i e_j e_k}{Q'}. \quad (2.16)$$

In particular, the metric itself is expressed in terms of the structure constants $g_{ij} = \tilde{C}_{ijq-1}$. In each particular case, i.e., for a given pair (p, q) , the structure constants can be directly

evaluated from the definition (2.16). For example, for $q = 5$, we find

$$\left| \begin{array}{c|c|c|c} \tilde{C}_1^{ij} & \tilde{C}_2^{ij} & \tilde{C}_3^{ij} & \tilde{C}_4^{ij} \\ \hline 5 & 0 & 0 & 0 \\ 0 & -3u_1 & -2u_2 & -u_3 \\ 0 & -2u_2 & -u_3 & 0 \\ 0 & -u_3 & 0 & 0 \end{array} \right| \left| \begin{array}{c|c|c|c} \tilde{C}_2^{ij} & \tilde{C}_3^{ij} & \tilde{C}_4^{ij} & \tilde{C}_5^{ij} \\ \hline 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & -2u_2 & -u_3 \\ 0 & 0 & -u_3 & 0 \end{array} \right| \left| \begin{array}{c|c|c|c} \tilde{C}_3^{ij} & \tilde{C}_4^{ij} & \tilde{C}_5^{ij} & \tilde{C}_6^{ij} \\ \hline 0 & 0 & 5 & 0 \\ 0 & 5 & 0 & 0 \\ 5 & 0 & 3u_1 & 0 \\ 0 & 0 & 0 & -u_3 \end{array} \right| \left| \begin{array}{c|c|c|c} \tilde{C}_4^{ij} & \tilde{C}_5^{ij} & \tilde{C}_6^{ij} & \tilde{C}_7^{ij} \\ \hline 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 5 & 0 & 3u_1 \\ 5 & 0 & 3u_1 & 2u_2 \end{array} \right|.$$

It is worth noting that such a simple linear dependence characterizes the structure constants in u coordinates only if two indices are raised. For all indices down, the dependence is nonlinear. This example demonstrates the following properties of the structure constants \tilde{C}_{ij}^k . For a given index i , it has a block diagonal structure, and regarded as a function of j and k , it depends on the combination $j + k$, which means that its counter-diagonal elements are the same. These properties together with the explicit calculations for particular values of the parameter q (one of which is shown above) suggest the general form of the structure constant in \mathbf{u} coordinates:

$$\tilde{C}_k^{ij} = (q + k - i - j + 1)u_{i+j-k-2}\tilde{\Theta}(i, j, k), \quad (2.17)$$

where $\tilde{\Theta}(i, j, k)$ is given by

$$\tilde{\Theta}(i, j, k) = \begin{cases} 1 & \text{if } i, j \leq k \text{ and } i + j > k, \\ -1 & \text{if } i, j > k \text{ and } i + j \leq k + q, \\ 0 & \text{otherwise.} \end{cases} \quad (2.18)$$

In (2.17), we imply that $u_{-1} = 1, u_0 = 0$. So far, we have not found a proof of the result (2.17). It therefore remains a conjecture verified up to $q = 15$.

The result (2.17) allows writing the partition function (2.8) in the form

$$\mathcal{F} = \frac{1}{2} \int_0^{\mathbf{u}^*} \tilde{\Omega}_i du^i, \quad (2.19)$$

where

$$\tilde{\Omega}_i = q \sum_{k,j}^{k+j=i+1} \frac{\partial S}{\partial u_k} \frac{\partial S}{\partial u_j} + \left[\sum_{l \geq i+3}^{2i} \sum_{k,j \leq i}^{k+j=l} - \sum_{l \geq 2i+2}^{i+q} \sum_{k,j > i}^{k+j=l} \right] (q + i - l + 1)u_{l-i-2} \frac{\partial S}{\partial u_k} \frac{\partial S}{\partial u_j}. \quad (2.20)$$

Hence, the found structure constants (2.17) together with the explicit form of the resonance relations, if it were known, could lead to a rather simple expression for the generating function of correlators. But one piece of data is missing: Which solution of the string equation is related to minimal models coupled to gravity? We discuss this question in the next section.

2.2 Flat coordinates

It was argued in [17] that only one of the solutions of the string equation is physical, i.e., it leads to zero one-point correlators. Hence, we just have to find the correct one. In the $q = 2$ case, there is only one solution (modulo some physically irrelevant phase factors). For $q = 3$, when there are two coordinates u_1 and u_2 , the situation is more subtle because the string equations have now a number of essentially different solutions. It was found in [13] that there exists only one particular solution with the necessary properties. The correctness of the choice was verified by computing the correlators. As a bonus, it turned out that the integration contour can be taken along one of two axes because $u_2^* = 0$ (for all couplings equal to zero) for this particular solution. We recall that the integral is independent of the contour because the integrand is a closed one-form.

For $q > 3$, the situation becomes more complicated. The computations show that unlike the $q = 2, 3$ cases, the roots of the string equation with only one nonzero component³ are nonphysical in the sense that using these roots in the expression (2.19) for the partition function, we cannot satisfy one of the basic requirements that one-point functions of physical fields are equal to zero. The consideration in the canonical coordinates now becomes more involved because the contour of the integration does not go along one of the axes. In brief, we conclude that the appropriate physical solution of the Douglas string equation corresponding to the general (p, q) MLG for an arbitrary value of the parameter q seems complicated in the canonical coordinates u_i .

Our idea is to find a transformation from the canonical coordinates to some new coordinates such that the physical solution would have the simplest form in the new coordinate system. We find that there exist coordinates v_i such that the physical solution in the zeroth order of the expansion in the coupling constants becomes $(v_1, 0, 0, \dots)$, i.e., $v_{k>1} = 0$, similarly to what happens in the canonical coordinates for $q = 2, 3$. Our calculations for different values of the parameter q led to the conclusion that the correct choice of the coordinates coincides with the choice of the flat coordinates on the Frobenius manifold. The explicit form of the transformation from the canonical to the flat coordinates is given by the equality

$$y = Q^{\frac{1}{q}}(y) - \frac{1}{q} \left(\frac{v_1}{Q^{\frac{1}{q}}(y)} + \frac{v_2}{Q^{\frac{2}{q}}(y)} + \dots + \frac{v_{q-1}}{Q^{\frac{q-1}{q}}(y)} \right) + \mathcal{O} \left(\frac{1}{Q^{\frac{q+1}{q}}(y)} \right), \quad (2.21)$$

where $Q(y)$ is defined in (2.5).

The flat coordinates have a number of useful properties. First, the metric in these coordinates is constant. It has the simple form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta, q}. \quad (2.22)$$

The flat coordinates are expressed explicitly in terms of u_i :

$$v_i = \theta_{i,0}. \quad (2.23)$$

³Here the couplings equal to zero.

Here, $\theta_{i,k}$ ($k \in \mathbb{N}$) is an important object of the Frobenius manifold structure which is relevant for the existence of infinite number of commuting flows and for the relation with the integrable hierarchies. (It represents a one-parameter family of deformed flat connections on the Frobenius manifold; see, e.g., [13] for more details). It is given by

$$\theta_{\alpha,k} = -c_{\alpha,k} \operatorname{res}_{y=\infty} Q^{k+\frac{\alpha}{q}}(y) \quad (2.24)$$

with

$$c_{\alpha,k}^{-1} = \left(\frac{\alpha}{q} \right)_{k+1}, \quad (2.25)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. The explicit form of the polynomial Q , evaluated in the flat coordinates on the solution of the string equation for the couplings equal to zero, is

$$Q(y)|_{\vec{v}_0} = y^q + q \sum_k^{2k-1 < q} \binom{q-k-1}{k-1} \left(\frac{v_1}{q} \right)^k y^{q-2k}. \quad (2.26)$$

This expression is exactly the one that appeared in the planar limit of the Q operator considered in [17]. The connection with the flat coordinates clarifies the geometric meaning of this choice of the Q operator.

Hence, we find that the string equation has one particular solution, denoted by \mathbf{v}^* in what follows, that is equal to $\{v_1^*, 0, 0, \dots\}$ in the zeroth order in coupling constants. It is shown in the next section that for this solution, the one-point correlation functions are zero and the two-point functions satisfy the diagonality requirement.

The structure constants in the flat coordinates are given by

$$C_{\alpha\beta\gamma} = -q \operatorname{res}_{y=\infty} \frac{\frac{\partial Q(y)}{\partial v^\alpha} \frac{\partial Q(y)}{\partial v^\beta} \frac{\partial Q(y)}{\partial v^\gamma}}{Q'(y)}. \quad (2.27)$$

In particular, because $C_{1\alpha\beta} = \eta_{\alpha\beta}$, we obtain

$$C_1^{\alpha\beta} = \delta_{\alpha+\beta,q} \quad \text{and} \quad C_\alpha^{q-1,\beta} = \delta_{\alpha,\beta}. \quad (2.28)$$

Finally, we note that because of the properties of the flat metric $\eta_{\alpha\beta}$, we obtain the useful relation

$$v^\alpha = v_{q-\alpha}. \quad (2.29)$$

Although we have obtained explicit structure constants in the canonical coordinates, we have not yet found the general answer for the structure constants in the flat coordinates. Nevertheless, taking the form of the generating function into account, we can note that to analyze one- and two-point correlation functions, we must know the structure constants in the flat coordinates only on the line $v_{k>1} = 0$. This becomes clear when we take into account that the contour independence of the expression (3.1) can be used to choose it along v_1 and also that for one- and two-point correlation functions, the derivatives act only on the integrand.

To find the structure constant, we perform the coordinate transformation with the expression (2.17)⁴

$$C_{\alpha\beta\gamma} = \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} \frac{\partial u^k}{\partial v^\gamma} \tilde{C}_{ijk}. \quad (2.30)$$

This allows obtaining an explicit form of the expansion in the vicinity of \mathbf{v}^* . We note that in the n -point correlation function, the contribution gives a maximum $(n-3)$ terms in this expansion. The explicit results for the first terms is presented below. To represent the answer compactly, we introduce the function $\Theta_{A,B}(x)$ such that $\Theta_{A,B}(x) = 1$ if $x \in [A, B]$ and is zero otherwise. The result for the structure constant in the flat coordinates on the line $v_{i>0} = 0$ is

$$C_{\alpha\beta\gamma} = \Theta_{1,q-1}(\alpha + \beta - \gamma) \left(-\frac{v_1}{q}\right)^{\frac{\alpha+\beta+\gamma-q-1}{2}} \text{ if } \frac{\alpha+\beta+\gamma-q-1}{2} \in \mathbb{N}, \text{ otherwise } 0. \quad (2.31)$$

In (2.31), we assume the ordering $\alpha \geq \beta \geq \gamma$. Because this tensor is symmetric, this information gives the complete answer. We give some details of the derivation in Appendix A.

3 Correlation functions

Our aim in this section is to determine the correlation functions in MLG using the special properties of the flat coordinates. All required information concerning these correlators is encoded in the partition function. In the flat variables,

$$\mathcal{F} = \frac{1}{2} \int_0^{\mathbf{v}^*} C_{\alpha}^{\beta\gamma} \frac{\partial S}{\partial v^\beta} \frac{\partial S}{\partial v^\gamma} dv^\alpha, \quad (3.1)$$

where S defined in (2.7) is now written in the flat coordinates. We use the resonance relations to satisfy the MLG selection rules. Explicitly, the resonance transformation is

$$\begin{aligned} t_{mn} &= \lambda_{mn} + \sum_{m_1, n_1}^{\delta_{m_1 n_1} \leq \delta_{mn}} A_{mn}^{m_1 n_1} \mu^{\delta_{mn} - \delta_{m_1 n_1}} \lambda_{m_1 n_1} + \\ &+ \sum_{m_1, n_1, m_2, n_2}^{\delta_{m_1 n_1} + \delta_{m_2 n_2} \leq \delta_{mn}} A_{mn}^{m_1 n_1, m_2 n_2} \mu^{\delta_{mn} - \delta_{m_1 n_1} - \delta_{m_2 n_2}} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \dots, \end{aligned} \quad (3.2)$$

where λ_{mn} are the Liouville coupling constants and the constants $A_{mn}^{m_1 n_1}, A_{mn}^{m_1 n_1, m_2 n_2}, \dots$ are to be defined from the conformal and fusion selection rules. We note that the specific property of the unitary series is the absence of resonance terms of the form μ^N . The gravitational dimensions δ_{mn} for the unitary MM $M(q+1, q)$ are given by

$$\delta_{mn} = \frac{2q+1 - |(q+1)m - qn|}{2q}. \quad (3.3)$$

⁴We use Greek and Latin alphabets for the respective \mathbf{v} and \mathbf{u} coordinate systems.

In particular, $\lambda_{11} = \mu$ is the cosmological constant, and $\lambda_{mn} \sim \mu^{\delta_{mn}}$. In what follows, λ_{mn} is used only for $(m, n) \neq (1, 1)$. By regrouping the terms, we can write the expansion of the action

$$S = S^{(0)} + \sum_{m,n} \lambda_{mn} S^{(mn)} + \sum_{m_1, n_1, m_2, n_2} \lambda_{m_1 n_1} \lambda_{m_2 n_2} S^{(m_1 n_1, m_2 n_2)} + \dots \quad (3.4)$$

The correlation numbers in the Liouville frame are related to the coefficients in the coupling constant decomposition of the partition function

$$\mathcal{F}[t(\lambda)] = Z_0 + \sum_{m_1, n_1} \lambda_{m_1 n_1} Z_{m_1 n_1} + \sum_{m_1, n_1} \sum_{m_2, n_2} \lambda_{m_1 n_1} \lambda_{m_2 n_2} Z_{m_1 n_1, m_2 n_2} + \dots \quad (3.5)$$

In other words, the correlation numbers are expressed in the form

$$\langle O_{m_1 n_1} \dots O_{m_N n_N} \rangle = Z_{m_1 n_1, \dots, m_N n_N} = \left. \frac{\partial}{\partial \lambda_{m_1 n_1}} \dots \frac{\partial}{\partial \lambda_{m_N n_N}} \right|_{\lambda=0} \mathcal{F}[t(\lambda)]. \quad (3.6)$$

One comment concerning dimensional analysis is in order. It is convenient to use their dimensionless counterparts instead of considering the dimensional quantities. This is achieved by changing variables $v_1 \rightarrow v_1/v_1^*$. In what follows, we let the same letter denote the new dimensionless variable. Omitted dimensional factors have the form of some powers of v_{*1} . They can be easily reconstructed using the homogeneity of the polynomial Q . These factors, of course, do not contribute to the real physical quantities, which are independent of the normalization.

3.1 One-point functions

In what follows, our basic tool is the recurrence relation [13]

$$\frac{\partial^2 \theta_{\lambda, k+1}}{\partial v_\alpha \partial v_\beta} = C_\gamma^{\alpha\beta} \frac{\partial \theta_{\lambda, k}}{\partial v_\gamma}. \quad (3.7)$$

The consideration is based on the following statement. On the line $v_{i>1} = 0$,

$$\begin{cases} k \text{ even} : & \frac{\partial \theta_{\lambda, k}}{\partial v_\alpha} = \delta_{\lambda, \alpha} x_{\lambda, k} \left(-\frac{v_1}{q} \right)^{\frac{k}{2}q}, \\ k \text{ odd} : & \frac{\partial \theta_{\lambda, k}}{\partial v_\alpha} = \delta_{\lambda, q-\alpha} y_{\lambda, k} \left(-\frac{v_1}{q} \right)^{\frac{k-1}{2}q+\lambda}, \end{cases} \quad (3.8)$$

where

$$x_{\lambda, k} = \frac{1}{\left(\frac{\lambda}{q}\right)_{\frac{k}{2}} \left(\frac{k}{2}\right)!} \quad \text{and} \quad y_{\lambda, k} = -\frac{1}{\left(\frac{\lambda}{q}\right)_{\frac{k+1}{2}} \left(\frac{k-1}{2}\right)!}. \quad (3.9)$$

The proof is given in Appendix B.

The first requirement to be satisfied is the zero expectation values of physical operators (except the unity operator). For the \mathbf{v}^* solution, we can take the contour of the integration

along the v_1 axis. Keeping in mind the comment at the end of the preceding section, we write the one-point function in the form

$$\langle O_{mn} \rangle = \int_0^1 C_{q-1}^{\beta\gamma} \frac{\partial S^{(0)}}{\partial v^\beta} \frac{\partial S^{(mn)}}{\partial v^\gamma} dv_1. \quad (3.10)$$

Because of the string equation, the derivative in (3.10) acts only on the integrand leading to $S^{(mn)}$, which is the first term of the series expansion of the action (3.4) in the coupling constants

$$S^{(mn)} = \operatorname{res}_{y=\infty} Q^{\frac{(q+1)m-qn}{q}} + \sum_{m_1, n_1}^{\delta_{m_1 n_1} < \delta_{mn}} A_{m_1 n_1}^{mn} \mu^{\delta_{mn} - \delta_{m_1 n_1}} \operatorname{res}_{y=\infty} Q^{\frac{(q+1)m_1 - qn_1}{q}}, \quad (3.11)$$

where $A_{m_1 n_1}^{mn}$ are the coefficients in the resonance relations of the coupling constants. Using (2.24), we find

$$S^{(0)} = \operatorname{res}_{y=\infty} \left[Q^{\frac{2q+1}{q}} + \mu Q^{\frac{1}{q}} \right] = -\frac{\theta_{1,2}}{c_{1,2}} - \mu \frac{\theta_{1,0}}{c_{1,0}}. \quad (3.12)$$

Taking (3.8) into account, we find that $\frac{\partial S^{(0)}}{\partial v^\beta} = 0$ for $\beta \neq q-1$. Hence,

$$\langle O_{mn} \rangle = \int_0^1 C_{q-1}^{q-1, \gamma} \frac{\partial S^{(0)}}{\partial v^{q-1}} \frac{\partial S^{(mn)}}{\partial v^\gamma} dv_1 = \int_0^1 \frac{\partial S^{(0)}}{\partial v_1} \frac{\partial S^{(mn)}}{\partial v_1} dv_1, \quad (3.13)$$

where we use $C_\alpha^{q-1, \beta} = \delta_{\alpha\beta}$ and lower the indices.

Now, the question is for which pairs (m, n) the factors $\frac{\partial S^{(mn)}}{\partial v_1}$ are nonzero. Using (3.8) and (3.11), we find that for $m-n = 2k+1$ with integer k , the corresponding gravitational dimension of the one-point function $\langle O_{mn} \rangle$ is

$$[\langle O_{mn} \rangle] = \frac{2q+1}{q} - \delta_{mn} = (k+1), \quad (3.14)$$

i.e., is analytic in μ and should not be considered [13]. The second option is $m-n$ even and $m=1$. Because $n \leq m$, the only possible pair is (m, n) , which corresponds to the unity operator.

It is instructive to obtain an explicit answer for the partition function

$$Z_0 = \frac{1}{2} \int_0^1 dv_1 \left(\frac{\partial S^{(0)}}{\partial v_1} \right)^2, \quad (3.15)$$

where

$$\frac{\partial S^{(0)}}{\partial v_1} = -\frac{1}{c_{1,2}} \frac{\partial \theta_{1,2}}{\partial v_1} - \frac{\mu}{c_{1,0}}. \quad (3.16)$$

The partial derivative $\frac{\partial \theta_{1,2}}{\partial v_1}$ can be easily found from (3.7):

$$\frac{\partial \theta_{1,2}}{\partial v_1} = \left(-\frac{1}{q} \right)^{q-2} \frac{v_1^q}{q}. \quad (3.17)$$

From the string equation $\left. \frac{\partial S^{(0)}}{\partial v_1} \right|_{v_1=v_1^*} = 0$, we find that the cosmological constant, measured in $(v_1^*)^q$, is

$$\mu = -q \left(-\frac{1}{q} \right)^q \frac{c_{1,0}}{c_{1,2}}. \quad (3.18)$$

The integral (3.15) can be evaluated explicitly:

$$Z_0 = \frac{\left(-\frac{1}{q}\right)^{2q-4}}{c_{1,2}^2} \frac{1}{(1+q)(1+2q)}. \quad (3.19)$$

We conclude that nonzero one-point functions appear only for the unity operator and for other operators having nonsingular gravitational dimensions. All singular one-point functions are automatically zero. We also note that when the flat variables are used on the level of one-point functions, the first order coefficients in the resonance expansion remain undetermined because the requirement of the absence of one-point expectation values is satisfied automatically. This is a specific property of the flat coordinate system. Hence, the explicit form of the resonance relation should be fixed from the restrictions arising on the levels of higher-point correlators.

3.2 Two-point functions

At the two-point level, the derivatives again act only on the integrand. Therefore, the result is

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle = \int_0^1 dv_1 C_{q-1}^{\alpha\beta} \frac{\partial S^{(m_1 n_1)}}{\partial v^\alpha} \frac{\partial S^{(m_2 n_2)}}{\partial v^\beta} + \int_0^1 dv_1 C_{q-1}^{\alpha\beta} \frac{\partial S^{(0)}}{\partial v^\alpha} \frac{\partial S^{(m_1 n_1, m_2 n_2)}}{\partial v^\beta}. \quad (3.20)$$

It follows from the same arguments as in the one-point case that the second integral vanishes. Hence,

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle = \int_0^1 dv_1 C_{q-1}^{\alpha\beta} \frac{\partial S^{(m_1 n_1)}}{\partial v^\alpha} \frac{\partial S^{(m_2 n_2)}}{\partial v^\beta}, \quad (3.21)$$

which because of (2.31) can be written as

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle = \sum_{\gamma=1}^{q-1} (-q)^{1-\gamma} \int_0^1 dv_1 v_1^{\gamma-1} \frac{\partial S^{(m_1 n_1)}}{\partial v_\gamma} \frac{\partial S^{(m_2 n_2)}}{\partial v_\gamma}. \quad (3.22)$$

Our next step is to derive the general form of the partial derivatives $\frac{\partial S^{(mn)}}{\partial v_\gamma}$. We find the following consequence of (3.8) for $\frac{\partial S^{(mn)}}{\partial v_\alpha}$:

$$\begin{cases} (m-n) \text{ even : } & \frac{\partial S^{(mn)}}{\partial v_\alpha} \sim \delta_{m,\alpha} \left(v_1^{\frac{m-n}{2}q} + \dots \right), \\ (m-n) \text{ odd : } & \frac{\partial S^{(mn)}}{\partial v_\alpha} \sim \delta_{m,q-\alpha} \left(v_1^{\frac{m-n-1}{2}q+m} + \dots \right), \end{cases} \quad (3.23)$$

where dots denote terms involving the cosmological constant μ . We now consider the two point functions (3.22). It follows from the dimensional analysis that

$$[\langle O_{m_1 n_1} O_{m_2 n_2} \rangle] = \frac{2q+1}{q} - \delta_{m_1 n_1} - \delta_{m_2 n_2} = \frac{m_1 - n_1}{2} + \frac{m_2 - n_2}{2} + \frac{m_1 + m_2}{2q}. \quad (3.24)$$

According to (3.22) and (3.23), there are three different cases. In the case where one of the fields is even while another is odd (for example, $m_1 - n_1$ is even and $m_2 - n_2$ is odd), we find that $m_1 + m_2 = q$, and the two-point function in this case is hence analytic in μ and is disregarded as nonuniversal.

If both differences $m_1 - n_1$ and $m_2 - n_2$ have the same parity, the dimensional analysis shows that

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle \sim \delta_{m_1, m_2} \mu^{\frac{m_1 - n_1}{2} + \frac{m_2 - n_2}{2} + \frac{m_1}{q}}. \quad (3.25)$$

Taking $m_1 \in [1, q-1]$ into account, we see that the two-point function cannot be analytic in μ and must therefore be equal to zero once the pairs (m_1, n_1) and (m_2, n_2) are different. It proves useful to introduce the new variable

$$t = 2 \left(\frac{v_1}{v_1^*} \right)^q - 1. \quad (3.26)$$

In the case where both fields are even, the degree of the polynomial $\frac{\partial S^{(mn)}}{\partial v_m}$ is

$$\frac{\partial S^{(mn)}}{\partial v_m}(t) \sim t^{\frac{m-n}{2}} + \dots, \quad (3.27)$$

where the dots denote terms subleading in t . The diagonality condition for the two-point correlation function in terms of the new variable takes the form

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle \sim \delta_{m_1, m_2} \int_{-1}^1 dt (1+t)^{\frac{m_1+m_2}{2q}-1} \frac{\partial S^{(m_1 n_1)}}{\partial v_{m_1}} \frac{\partial S^{(m_2 n_2)}}{\partial v_{m_2}} = 0. \quad (3.28)$$

In the case where both fields are odd, it is convenient to change $\frac{\partial S^{(mn)}}{\partial v_\alpha} = (1+t)^{\frac{m}{q}} \frac{\partial \tilde{S}^{(mn)}}{\partial v_\alpha}$. The degree of the new polynomial is

$$\frac{\partial \tilde{S}^{(mn)}}{\partial v_\alpha} \sim t^{\frac{m-n-1}{2}} + \dots, \quad (3.29)$$

and the diagonality condition is

$$\langle O_{m_1 n_1} O_{m_2 n_2} \rangle \sim \delta_{m_1, m_2} \int_{-1}^1 dt (1+t)^{\frac{m_1+m_2}{2q}} \frac{\partial \tilde{S}^{(m_1 n_1)}}{\partial v_{q-m_1}} \frac{\partial \tilde{S}^{(m_2 n_2)}}{\partial v_{q-m_2}} = 0. \quad (3.30)$$

Equations (3.28) and (3.30) show that the diagonal form of the two-point correlation numbers requires that the polynomials (3.27) and (3.29) form an orthogonal set of polynomials. Such polynomials are known as Jacobi orthogonal polynomials. In our case we are dealing with the special class of the Jacobi polynomials $P_n^{(0,b)}(t)$. They are n th order

polynomials that form an orthogonal system on the interval $[0, 1]$ with the orthogonality condition

$$\int_{-1}^1 dt (1+t)^b P_n^{(0,b)}(t) P_m^{(0,b)}(t) = \frac{2^{b+1}}{2n+b+1} \delta_{m,n}. \quad (3.31)$$

In the standard normalization, $P_n^{(0,b)}(1) = 1$. We note that the Jacobi polynomials in this normalization have the highest coefficient

$$P_n^{(0,b)}(t) = \frac{(b+n+1)_n}{n!} \left(\frac{t}{2}\right)^n + \dots \quad (3.32)$$

Summarizing these results, we conclude that

$$\frac{\partial S^{(mn)}}{\partial v_\alpha}(v_1) = \begin{cases} \delta_{m,\alpha} v_{*1}^{\frac{m-n}{2}q} N_{mn} P_{\frac{m-n}{2}}^{(0, \frac{m-q}{q})}(t), & (m-n) \text{ even}, \\ \delta_{m,q-\alpha} v_{*1}^{\frac{m-n-1}{2}q+m} N_{mn} (1+t)^{\frac{m}{q}} P_{\frac{m-n-1}{2}}^{(0, \frac{m}{q})}(t), & (m-n) \text{ odd}, \end{cases} \quad (3.33)$$

where t is defined in (3.26) and N_{mn} is the normalization factor independent of t . Although we obtained (3.33) for $v_{i>1} = 0$, the general form of $S^{(mn)}$ can be easily reconstructed using equation (3.11). Hence, we have the answer for the action S defined in (3.4) up to second order in the coupling constant expansion. This result is required for calculating multipoint correlation functions.

4 Conclusions

In this paper, we presented the solution of the Douglas string equation and the explicit form of the resonance relations up to second order in the coupling constant expansion, which lead to satisfying the main requirements of minimal Liouville gravity, namely, the requirements that the one-point correlation function except the unity operator must be zero and that the two-point correlators must be diagonal. Using the connection between the Douglas approach and the Frobenius manifold structure, we showed that the appropriate solution of the string equation has a remarkably simple form in the flat coordinates. Calculations of the multipoint correlation functions require this information. In addition, it requires knowing the structure constants of the Frobenius algebra. We found the structure constants in the canonical coordinates as well as the expression for the structure constants on the solution of the string equation in the flat coordinates, i.e., for all except the first coordinates be zero. To obtain the general answer for the partition function, which is appropriate for calculating arbitrary multipoint correlators, the explicit form of the general structure constants in the flat coordinates is required. The general form of the resonance relations is also needed. Presumably, higher terms in the coupling constant decomposition are fixed from the restrictions following from the fusion rules. We are going to study this question in the near future.

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Appendix A $C_{\alpha\beta\gamma}$

If two of the indices are raised, then the structure constants in \mathbf{u} coordinates is known from (2.17). On the other hand, the metric in flat coordinates is simple, and lowering an index α hence is just replacing it with $q - \alpha$. Therefore,

$$C_{\alpha\beta\gamma} = \frac{\partial v^{q-\alpha}}{\partial u^i} \frac{\partial v^{q-\beta}}{\partial u^j} \frac{\partial u^k}{\partial v^\gamma} \tilde{C}_k^{ij}. \quad (\text{A.1})$$

Using the explicit form of the coordinate transformation (2.21), we find that on the line $v_{k>1} = 0$, we have

$$\frac{\partial u^k}{\partial v^\gamma}(\mathbf{v}) = U_\gamma^k, \quad (\text{A.2})$$

$$\frac{\partial v^\alpha}{\partial u^i}(\mathbf{v}) = V_\gamma^k, \quad (\text{A.3})$$

where

$$U_\gamma^k = \begin{pmatrix} \frac{\gamma-k+q-2}{2} \\ \frac{\gamma+k-q}{2} \end{pmatrix} \left(\frac{v_1}{2} \right)^{\frac{\gamma+k-q}{2}} \quad \text{if} \quad \frac{\gamma+k-q}{2} \in \mathbb{N}, \quad \text{otherwise } 0, \quad (\text{A.4})$$

$$V_j^\rho = \frac{2\rho}{q+\rho-j} \begin{pmatrix} q-j-1 \\ \frac{q-\rho-j}{2} \end{pmatrix} \left(-\frac{v_1}{2} \right)^{\frac{q-\rho-j}{2}} \quad \text{if} \quad \frac{q-\rho-j}{2} \in \mathbb{N}, \quad \text{otherwise } 0. \quad (\text{A.5})$$

In particular, from (A.4), we find

$$u^k = \frac{2q}{k+1} \begin{pmatrix} \frac{2q-k-1}{2} \\ \frac{k-1}{2} \end{pmatrix} \left(\frac{v_1}{2} \right)^{\frac{k+1}{2}} \quad \text{if} \quad \frac{k+1}{2} \in \mathbb{N}, \quad \text{otherwise } 0. \quad (\text{A.6})$$

In this notation, we find

$$C_{\alpha\beta\gamma}(\mathbf{v}_*) = V_j^{q-\alpha} V_j^{q-\beta} U_\gamma^k \tilde{C}_k^{ij}. \quad (\text{A.7})$$

Using standard identities for the binomial coefficients, we obtain (2.31).

Appendix B Evaluation of $\frac{\partial \theta_{\lambda,k}}{\partial v_\alpha}$

We use our basic recurrence relation

$$\frac{\partial^2 \theta_{\lambda,k}}{\partial v_1 \partial v_\alpha} = C_\gamma^{1\alpha} \frac{\partial \theta_{\lambda,k-1}}{\partial v_\gamma}. \quad (\text{B.1})$$

We integrate this equation over v_1 to find

$$\frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \int^{v_1} C_\gamma^{1\alpha} \frac{\partial \theta_{\lambda,k-1}}{\partial v_\gamma} dv_1, \quad (\text{B.2})$$

where the integration constant is absent because it is independent of v_1 and can hence depend on only $v_{\alpha>1}$, which are set to zero. Using this equation iteratively, we find

$$\frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \int^{v_1} dv_1^{(1)} \dots \int^{v_1^{(k-1)}} dv_1^{(k)} C_{\gamma_1}^{1\alpha}(v_1^{(1)}) C_{\gamma_2}^{1\gamma_1}(v_1^{(2)}) \dots C_{\gamma_k}^{1\gamma_{k-1}}(v_1^{(k)}) \frac{\partial \theta_{\lambda,0}}{\partial v_{\gamma_k}}, \quad (\text{B.3})$$

where explicitly $\frac{\partial \theta_{\lambda,0}}{\partial v_{\gamma_k}} = \delta_{\lambda,\gamma_k}$. From the explicit form of the structure constant (2.31), we conclude that

$$\gamma_1 = \gamma_3 = \dots = q - \alpha, \quad (\text{B.4})$$

$$\gamma_2 = \gamma_4 = \dots = \alpha. \quad (\text{B.5})$$

In particular, we find that $\lambda = \alpha$ for even k while $\lambda = q - \alpha$ for odd k . Otherwise, the integral is equal to zero. After contracting all the indices, we are left with a multiple integral containing

$$C_{q-\alpha}^{1\alpha}(v_1^{(i)}) = \left(-\frac{v_1^{(i)}}{q}\right)^{q-\alpha-1} \quad (\text{B.6})$$

and

$$C_\alpha^{1q-\alpha}(v_1^{(i)}) = \left(-\frac{v_1^{(i)}}{q}\right)^{\alpha-1}. \quad (\text{B.7})$$

Explicit integration gives

$$\begin{cases} k \text{ even : } & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,\alpha} \left(-\frac{1}{q}\right)^{\frac{k}{2}(q-2)} \frac{v_1^{\frac{k}{2}q}}{\alpha q(\alpha+q)2q(\alpha+2q)3q\dots(\alpha+(\frac{k}{2}-1)q)(\frac{k}{2}q)}, \\ k \text{ odd : } & \frac{\partial \theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,q-\alpha} \left(-\frac{1}{q}\right)^{\frac{k-1}{2}(q-2)+q-\alpha-1} \frac{v_1^{\frac{k+1}{2}q-\alpha}}{(q-\alpha)q(2q-\alpha)2q(3q-\alpha)3q\dots(\frac{k-1}{2}q)(\frac{k+1}{2}q-\alpha)}. \end{cases} \quad (\text{B.8})$$

This gives (3.8) and (3.9).

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